# METHOD OF OUTER AND INNER ASYMPTOTIC EXPANSIONS IN THE THEORY OF BROWNIAN MOTION OF AEROSOL PARTICLES

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We examine the Brownian motion of particles in a gaseous medium, complicated by the influence of inertial forces. The equation for the distribution function in phase space describing motion of this type was obtained in [1]. Also presented in [1] are the solutions of this equation for certain simple particular cases. The approximate equations of motion of aerosol particles in coordinate space were first obtained in [2] and solved for certain concrete problems in [3, 4]. More exact equations of motion in coordinate space, and also the limits of applicability of the equations of [2], are presented in [5].

1. The Fokker–Planck equation for the distribution function of aerosol particles, obtained in [1], has the form

$$k\frac{\partial f}{\partial t}+(k\nabla-\nabla_0\cdot\mathbf{v}')f+(\mathbf{u}+\mathbf{F}\cdot\nabla_0)f=\frac{1}{k\lambda}\nabla_0{}^2f$$
$$(k=mu_\infty/bl,\quad \lambda=mu_\infty{}^2/kKT) \tag{1.1}$$

Here v' is the velocity of the particles in the system; u is the velocity of the gaseous medium; F is the dimensionless external force acting on the particles (measured in $bu_\infty$ units); $\nabla$ and $\nabla_0$ are the Hamilton operators (coordinate and velocity, respectively); k and $\lambda$ are the Stokes and Peclet numbers, respectively; m is the particle mass; b is a constant of proportionality between the resistance force and the relative velocity of the particle in the medium; $u_\infty$ and $l$ are the characteristic flow velocity and dimension; K is Boltzmann's constant; T is absolute temperature. We assume that

$$b=\text{const},\qquad T=\text{const}$$

Assume that at some instant local equilibrium, described by the following distribution function, occurs in the system:

$$f_0=\left(\frac{k\lambda}{2\pi}\right)^{3/2}n\exp\left(-\frac{1}{2}k\lambda\,|\mathbf{v}'-\mathbf{v}|^2\right)\qquad(t=t_0) \tag{1.2}$$

Here v(r, t) and n(r, t) are the average velocity and concentration. This is possible, for example, in uniform translational flow if $t_0$ exceeds significantly the relaxation time of the aerosol particles. If the first derivatives of v are small, the solution of (1.1) with the initial condition (1.2) can be written approximately in the form

$$f\approx f_0\qquad(t>t_0) \tag{1.3}$$

In fact, direct substitution of (1.2) into (1.1) shows that (1.3) would be exact if the right-hand side of (1.1) had the form

$$\frac{1}{k\lambda}(\nabla_0{}^2+kD_{ij}\nabla_{0i}\nabla_{0j})f\qquad\left(2D_{ij}=\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \tag{1.4}$$

With the aid of (1.3) we obtain from (1.1) the following equations for n and v [5]

$$k\left[\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right]\mathbf{v}+\mathbf{v}=\mathbf{u}+\mathbf{F}-\frac{1}{\lambda}\nabla\ln n \tag{1.5}$$

$$\frac{\partial n}{\partial t}+\operatorname{div}n\mathbf{v}=0 \tag{1.6}$$

Thus, on the average the motion of an ensemble of Brownian aerosol particles can be identified with the motion of some compressible continuum; hereafter we call this abstract medium the aerosol fluid.

Equation (1.6) can be transformed to the Smoluchowski convective diffusion equation

$$\frac{\partial n}{\partial t}+\operatorname{div}n\mathbf{v}^*=\frac{1}{\lambda}\nabla^2n\qquad\left(\mathbf{v}^*=\mathbf{v}+\frac{1}{\lambda}\nabla\ln n\right) \tag{1.7}$$

It is natural to call the vector $\mathbf{v}^*$ the convective velocity of the aerosol fluid. From (1.5) we have

$$k\left[\frac{\partial}{\partial t}+\left(\mathbf{v}^*-\frac{1}{\lambda}\nabla \ln n\cdot\nabla\right)\right]\left(\mathbf{v}^*-\frac{1}{\lambda}\nabla \ln n\right)+\mathbf{v}^*=\mathbf{u}+\mathbf{F} \tag{1.8}$$

2. Let us assume that there is an obstacle (body) in the uniform translational stream. We take $u_\infty$ as the velocity of the undisturbed flow; $l$ is the radius of the body cross section. During precipitation of aerosol particles onto the body under normal conditions the most effective interaction between the diffusive and inertial motions would obviously be expected if the particle dimension $\gamma_0$ is on the order of $10^{-5}$ cm. Let the density of the matter in the particle be approximately 1 $\mathrm{g\cdot cm^{-3}}$. Then

$$b\sim 10^{-8}\,\mathrm{g\cdot sec^{-1}},\quad k\sim 10^{-7}\,u_\infty/l,\quad \lambda\sim 10^{6}\,u_\infty l$$
$$([u_\infty]=\mathrm{cm/sec},\ [l]=\mathrm{cm}) \tag{2.1}$$

It is natural to assume that $l \geqslant 10^{-3}$ cm and $u_\infty \leqslant 10^4$ cm/sec. Thus, the most effective interaction between the diffusive and inertial motions will occur for

$$k\sim 1,\qquad \lambda\gg 1 \tag{2.2}$$

It follows from these relations that the influence of particle inertia on their Brownian diffusion need be studied only for very large $\lambda$.

It is convenient to study (1.7) and (1.8) in the $O\xi\eta$ orthogonal curvilinear coordinates (for simplicity we consider only the plane case). We denote the surface of the obstacle-body by $\Gamma$. We assume that the surface $\Gamma$ is singly connected; at each point of the surface we can draw only a single normal; these normals do not cross anywhere in the outer layer P adjacent to $\Gamma$; the thickness of the P layer is nowhere equal to zero; on $\Gamma$ there is a single forward flow stagnation point (we denote it by the letter N). Then the $O\xi\eta$ coordinate system, whose $O\xi$-axis is directed along the normal to $\Gamma$ and $O\eta$-axis is parallel to $\Gamma$, will be nondegenerate in P. From purely geometric arguments we can establish that

$$d\mathbf{r}=\mathbf{e}_\xi d\xi+\mathbf{e}_\eta\frac{R+\xi}{R}d\eta$$
$$\nabla=\mathbf{e}_\xi\frac{\partial}{\partial\xi}+\mathbf{e}_\eta\frac{R}{R+\xi}\frac{\partial}{\partial\eta}$$
$$\frac{\partial\mathbf{e}_\xi}{\partial\xi}=\frac{\partial\mathbf{e}_\eta}{\partial\xi}=0,\quad\frac{\partial\mathbf{e}_\eta}{\partial\eta}=-\frac{1}{R}\mathbf{e}_\xi,\quad\frac{\partial\mathbf{e}_\xi}{\partial\eta}=\frac{1}{R}\mathbf{e}_\eta$$

Here $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ are the unit vectors of the $O\xi\eta$ system, R is the radius of $\Gamma$. In the $O\xi\eta$ system (1.7) and (1.8) take the form

$$\begin{aligned}&\frac{\partial n}{\partial t}+\frac{\partial}{\partial\xi}nv_\xi^*+\frac{1}{R+\xi}nv_\xi^*+\frac{R}{R+\xi}\frac{\partial}{\partial\eta}nv_\eta^*\\&=\frac{1}{\lambda}\frac{R}{R+\xi}\left\{\frac{\partial}{\partial\xi}\frac{R+\xi}{R}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\right\}n\end{aligned} \tag{2.3}$$

$$\begin{aligned}k\Big[&\frac{\partial}{\partial t}\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial\ln n}{\partial\xi}\right)+\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\ln n\right)\frac{\partial}{\partial\xi}\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\ln n\right)\\&+\frac{R}{R+\xi}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)\frac{\partial}{\partial\eta}\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\ln n\right)\\&-\frac{1}{R+\xi}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)^2\Big]+v_\xi^*=u_\xi+F_\xi\end{aligned} \tag{2.4}$$

$$\begin{aligned}k\Big[&\frac{\partial}{\partial t}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)+\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\ln n\right)\frac{\partial}{\partial\xi}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)\\&+\frac{R}{R+\xi}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)\frac{\partial}{\partial\eta}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)\\&+\frac{1}{R+\xi}\left(v_\eta^*-\frac{1}{\lambda}\frac{R}{R+\xi}\frac{\partial}{\partial\eta}\ln n\right)\left(v_\xi^*-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\ln n\right)\Big]+v_\eta^*=u_\eta+F_\eta\end{aligned} \tag{2.5}$$

The formulation of the boundary conditions for these equations in the general case is not trivial; however, we do not dwell on this here. It is natural to take [2]

$$\mathbf{v}^*\to\mathbf{u}+\mathbf{F},\quad n\to 1\quad\text{as}\quad\xi\to\infty \tag{2.6}$$

$$\frac{\partial n}{\partial\xi}=a(n-n_0)\quad\text{as}\quad\xi\to 0 \tag{2.7}$$

Here $a$ and $n_0$ are constants which are determined from some physical considerations or other. The simplest case for (2.3)–(2.5) is as $a \to \infty$ and $n_0 \to 0$. Then the boundary condition on $\Gamma$ takes the form

$$n = 0 \quad \text{as} \quad \xi \to 0 \tag{2.8}$$

The asymptotic behavior of $v^*$ and n for large $\lambda$ depends significantly on the vectors **u** and F in the vicinity of $\Gamma$. For viscous flow, essentially,

$$u_\xi \approx u_1 \xi^2, \qquad u_\eta \approx u_2 \xi \qquad (\xi \to 0) \tag{2.9}$$

If the vector **u** is solenoidal, we can introduce the flow stream function $\psi$. By virtue of (2.9) the stream function in the vicinity of $\Gamma$ is approximately equal to

$$\psi \approx \psi_0 \xi^2, \qquad \psi_0 > 0 \qquad (\xi \to 0) \tag{2.10}$$

For potential flow past the body the asymptotic behavior of **u** in the vicinity of $\Gamma$ is not described by (2.9). However, there is no need to dwell on the potential flow case here. It makes sense to consider diffusion of the aerosol particles to the body only for those k for which there is no purely inertial particle flux to this body, i. e., for $k < k^*$, where $k^*$ is the critical value of the Stokes number. In fact, for $k > k^*$ the aerosol particle flux to the body due to particle inertia increases very rapidly with increase of k and begins to exceed considerably the diffusive flux. It was shown in [6] that the particle size for $k \sim k^*$ is two orders of magnitude less than the thickness of the hydrodynamic boundary layer in the vicinity of the point N for arbitrarily large values of the flow Reynolds number. Therefore in the calculations of particle diffusion we must of necessity take into account the influence of the hydrodynamic boundary layer; the use of the purely potential model for flow past the body for $k \leqslant k^*$ is physically meaningless.

In order to write formulas of the type (2.9) for the vector **F** we must concretize the physical nature of the interaction force between the particles and the body. For electrostatic attraction, for example,

$$F_\xi \approx \alpha\,(-a_0 + a_1\xi + a_2\xi^2), \qquad F_\eta = 0 \qquad (\xi \to 0)$$
$$(a_0 > 0,\ \alpha > 0) \tag{2.11}$$

Here $\alpha$ is a parameter characterizing the degree of interaction.

We neglect hydrodynamic interaction of the particles with the body and the finite particle size, which leads to the so-called "anchoring effect." Analysis of the influence of these two factors on the diffusion is so complicated that it deserves separate study.

3. We divide the flow region arbitrarily into three parts: $Q_0$ is the region adjacent to the base part of the surface $\Gamma$ and extends downstream, $Q_1$ is a thin layer adjacent to $\Gamma$ in the forward and side areas, and $Q_2$ is the remaining part of the flow. The gradients of the concentration n in $Q_2$ are very small; therefore, for very large $\lambda$ the diffusion terms in (2.3)–(2.5) can be neglected. As a result we obtain

$$k\frac{\partial \mathbf{v}}{\partial t} + k(\mathbf{v}\nabla)\mathbf{v} + \mathbf{v} \approx \mathbf{u} + \mathbf{F}, \qquad \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} \approx 0 \tag{3.1}$$

In the stationary case with $\mathbf{F} = \mathbf{F}$ the behavior of the solutions of the first equation in (3.1) has been studied in detail in [6]. We write the asymptotic formulas obtained in [6] as $\xi \to 0$,

$$\mathbf{v} = \text{const} \quad \text{for} \quad k > k^* \tag{3.2}$$

$$v_\xi \approx \left(u_1 + \frac{k}{R}u_2^2\right)\xi^2, \quad v_\eta \approx u_2\xi \quad \text{for} \quad k < k^* \tag{3.3}$$

We denote the value of n at the point N by $n_N$. To determine $n_N$ we must solve the problem

$$k(\mathbf{v}\nabla)\mathbf{v} + \mathbf{v} = \mathbf{u}, \qquad \operatorname{div} n\mathbf{v} = 0$$
$$\mathbf{v} \to \mathbf{u}, \qquad n \to 1 \quad \text{for} \quad \xi \to \infty \tag{3.4}$$

Problem (3.4) can only be solved numerically. Finding from (3.3) div **v** and substituting this value into the second equation in (3.4), we obtain for $k < k^*$

$$n \approx n_N \exp\left(-2k\int_0^\eta u_2 \frac{d\eta}{R}\right) \qquad (\xi \to 0) \tag{3.5}$$

Formulas (3.3) and (3.5) are independent of the boundary condition for $\mathbf{v}$ as $\xi \to \infty$. This leads to more restrictive conditions for their applicability in comparison with the applicability conditions for (2.9).

Assume that relations (2.9) are effective for $\xi \leqslant \xi_0$ and the exact solution of (3.4) has been found for $\xi = \xi_0$. Then in calculating $\mathbf{v}$ and n in the region $0 \leq \xi < \xi_0$ we can use the boundary condition

$$n = n^{(0)}, \qquad \mathbf{v} = \mathbf{v}^{(0)} \quad \text{for } \xi = \xi_0$$

It is easy to show that the size of the region of influence of these boundary conditions is proportional to k. With increase of k the region of influence increases and for $k = k^*$ becomes equal to $\xi_0$. **Consequently, the necessary** condition for the applicability of (3.3) and (3.5) will have the form

$$\xi \lesssim \xi_0 (1 - k / k^*) \tag{3.6}$$

Formulas (3.3) and (3.5) are asymptotic expansions in $\xi$ of the principal terms of the outer solution of (2.3)–(2.5) for $\mathbf{F} = 0$ and $k < k^*$.

4. Adaptation of the concentration n to the boundary value at the surface $\Gamma$ in the general case is possible if and only if in the transition zone between the regions $Q_2$ and $Q_1$ **the diffusive transport to $\Gamma$ is comparable with the** convective transport, no matter how large $\lambda$. Naturally, with increase of $\lambda$ this transition zone will be ever closer to the surface $\Gamma$, and therefore there must be some $\lambda^*$ such that for $\lambda < \lambda^*$ in the congruence condition we can substitute in place of $\mathbf{V}$ the principal term of the asymptotic expansion in $\xi$. Thus,

$$n\left(u_1 + \frac{k}{R} u_2^2\right) \xi^2 \sim \frac{1}{\lambda} \frac{\partial n}{\partial \xi} \qquad (k < k^*,\ \lambda \to \infty) \tag{4.1}$$

Condition (4.1) can be satisfied only with a very sharp change of the concentration in the direction toward $\Gamma$. Let us assume that there exists a coordinate system $\{\zeta, \eta\}$ in which n will vary more or less smoothly in the direction toward $\Gamma$, no matter how large $\lambda$:

$$\partial n / \partial \zeta \sim 1 \qquad (\lambda \to \infty) \tag{4.2}$$

Let us further assume that the connection between $\xi$ and $\zeta$ is linear. Then we find from (4.1)

$$\zeta = \lambda^{1/3} \xi \tag{4.3}$$

The validity of these assumptions is confirmed by direct analysis of the solutions of (2.3)–(2.5) obtained with their use.

For comparison we note that for $k > k^*$ similar arguments lead to the formula

$$\zeta = \lambda \xi \tag{4.4}$$

Using (4.2) and (4.3), we find from (2.3)–(2.5) in the steady-state case

$$v_\xi^* \approx \left(u_1 + \frac{k}{R} u_2^2\right) \zeta^2 \lambda^{-2/3} \tag{4.5}$$

$$v_\eta^* \approx u_2 \zeta \lambda^{-1/3} \tag{4.6}$$

$$\zeta^2 \left(u_1 + \frac{k}{R} u_2^2\right) \frac{\partial n}{\partial \zeta} + \zeta u_2 \frac{\partial n}{\partial \eta} + 2k \frac{u_2^2}{R} \zeta n \approx \frac{\partial^2 n}{\partial \zeta^2} \tag{4.7}$$

In accordance with the principle of asymptotic matching of the outer and inner expansions [7], the solutions of (4.5)–(4.7) as $\zeta \to \infty$ must approach the solutions of (3.4), written in the new variables. It is easy to see that the **solutions** of (4.5) and (4.6) automatically match, and the solution of (4.7) will match provided

$$n \to n_N \exp\left(-2k \int_0^\eta u_2 \frac{d\eta}{R}\right) \qquad (\zeta \to \infty) \tag{4.8}$$

This relation must then be taken as the boundary condition for (4.7). It is worthy of note that for the boundary conditions (2.8) and (4.8) the solution of (4.7) can be written in closed analytic form

$$n \approx n_N \exp\left(-4k\int_0^{\eta}\psi_0 \frac{d\eta}{R}\right)\frac{\gamma(^1/_3, \zeta^3/\mu)}{\Gamma(^1/_3)}$$

$$\left(\mu = \frac{9}{2}\psi_0^{-3/2}\int_0^{\eta} d\eta'\psi_0^{1/2}\exp\left(6k\int_{\eta'}^{\eta}\psi_0\frac{d\eta}{R}\right)\right) \tag{4.9}$$

Here $\gamma$ is the incomplete gamma function. This solution is an extension of the solutions found for certain concrete bodies in [3, 8].

Formulas (4.5), (4.6), and (4.9) are the principal terms of the inner solution. If the next terms of the expansions in $\xi$ are of order $\xi^3$, $\xi^2$ and $\xi$ (for $\nu_\xi$, $\nu_\eta$, and n, respectively), then the terms dropped in (4.5), (4.6), and (4.9) are of order $\lambda^{-1}$, $\lambda^{-2/3}$, and $\lambda^{-1/3}$ in $\lambda$. The outer solution can always be represented in series form,

$$\mathbf{v} \approx \sum_{i\geqslant 0}\mathbf{v}^{(i)}\lambda^{-i}, \qquad n \approx \sum_{i\geqslant 0} n^{(i)}\lambda^{-i} \qquad (\lambda\to\infty) \tag{4.10}$$

We assume that

$$\mathbf{v}^{(i)} \approx \sum_{m\geqslant 0}\mathbf{v}_m^{(i)}\xi^{m+1}, \quad n^{(i)} \approx \sum_{m\geqslant 0} n_m^{(i)}\xi^m, \quad v_{0\xi}^{(i)} = 0 \quad (\xi\to 0) \tag{4.11}$$

Then the inner solution is written in the form

$$\mathbf{v}^* \approx \sum_{i\geqslant 0}\mathbf{v}_i\lambda^{-i/3-1/3}, \quad n \approx \sum_{i\geqslant 0} n_i\lambda^{-i/3}, \quad v_{0\xi} = 0 \quad (\lambda\to\infty) \tag{4.12}$$

The correctness of (4.10) is verified directly by substituting these series into (2.3)–(2.5), written in the variables $\{\zeta, \eta\}$, and analysis of the matching conditions constructed on the basis of (4.10)–(4.11).

Formulas (4.5)–(4.8) retain the same form for axisymmetric problems as well. In connection with the conversion to the stream function, (4.9) changes somewhat in the axisymmetric case. We denote by $R_1$ the distance from the axis of symmetry to the point on the surface $\Gamma$ with the coordinate $\eta$ (here $\eta$ is the arc length of the contour which arises when the surface $\Gamma$ is intersected by any plane passing through the axis of symmetry). Then

$$n \approx n_N \exp\left\{-4k\int_0^{\eta}\psi_0\frac{d\eta}{R_1}\right\}\frac{\gamma(^1/_3, \zeta^3/\mu)}{\Gamma(^1/_3)}$$

$$\left(\mu = \frac{9}{2}\psi_0^{-3/2}\int_0^{\eta} d\eta'\,\frac{R_1}{R}\psi_0^{1/2}\exp\left(6k\int_{\eta'}^{\eta}\psi_0\frac{d\eta}{RR_1}\right)\right) \tag{4.13}$$

5. The internal solution obtained is physically meaningful only when the thickness of the $Q_1$ layer exceeds considerably the dimension of the aerosol particles. This condition defines the upper limit with regard to $\lambda$ of the effectiveness of (4.9) and (4.13). The thickness of the $Q_1$ layer is of order

$$\delta \sim \mu^{1/3}\lambda^{-1/3} \tag{5.1}$$

Thus, (4.9) and (4.13) are effective for

$$\lambda \lesssim l^3/\gamma_0^3 \tag{5.2}$$

For very large flow Reynolds numbers the hydrodynamic boundary layer thickness is of order

$$\delta^* \sim \nu^{1/2}/u_\infty^{1/2} l^{1/2}$$

Naturally, the solutions (4.5), (4.6), (4.9), and (4.13) are applicable if $\delta \ll \delta^*$, i. e., for particles whose size satisfies the inequality

$$\gamma_0 \gg 10^{-10} u_\infty^{1/2} l^{1/2}/\nu^{1/2}\ \mathrm{cm} \tag{5.3}$$

For particles whose size does not satisfy this condition, at large flow Reynolds numbers it is necessary to solve an equation of the type (4.7) with coefficients which depend in a complex fashion on $\zeta$.

It is easy to establish that with increase of $\eta$, $\mu$ **varies from** some finite value up to $\infty$. Therefore the thickness of $Q_1$ is minimum in the vicinity of N; with increase of $\eta$, $\mu$ increases and takes infinitely large values in the base region. Comparing (3.6) and (5.1), we conclude that (4.5), (4.6), (4.9), and (4.13) are applicable only for those $\eta$ for which

$$\mu \lesssim \lambda\xi_0^3 (1 - k / k^*)^3 \tag{5.4}$$

It is obvious that in the base flow region condition (5.4) will not be satisfied. In $Q_1$ the convective particle transport is also comparable with the diffusive transport although, in contrast with $Q_1$, in directions perpendicular to the motion as $\xi \to \infty$. Unfortunately this condition does not lead to as significant a simplification of the equations as in region $Q_1$.

Differentiating (4.9) with respect to $\zeta$, we find

$$j \approx \frac{n_N}{\Gamma(1+1/3)}\mu^{-1/3} \exp\left(-2k\int_0^\eta u_2 \frac{d\eta}{R}\right)\lambda^{-2/3} \tag{5.5}$$

It can be shown that $n_N$ always increases with increase of k, and significantly so. For example, in the case of Stokes flow past a sphere [3] for $k/k^* \approx 0.164$, $n_N$ equals 1.85, for $k/k^* \approx 0.328$, $n_N$ **already becomes equal to 3.01, and for** $k/k^* = 1$, $n_N$ equals infinity for all bodies. Thus the factor $n_N$ in (3.5) leads to increase of the diffusional particle flux to the body with increase of k. The centrifugal forces, described in (5.5) by the exponential terms, reduce the diffusional flux to the side portions of the body. Thus, the inertia of the aerosol particles leads to considerable deformation of the form of the diffusional precipitation on the body. It is difficult to establish in the general case which terms make the principal contribution to the integral particle flux to the body (for the sphere in Stokes flow the integral flux varies as $(1-0.480\, k)$ for small k).

Integrating (5.5) over the surface $\Gamma$, we find that

$$I = \frac{3^{1/3}}{2^{2/3}\Gamma(1+1/3)} n_N \left[\int_\Gamma d\Gamma\, \psi_0^{1/2} \exp\left(-6k\int_0^\eta \psi_0 \frac{d\eta}{R}\right)\right]^{2/3} \lambda^{-2/3} + O(\lambda^{-1}) \tag{5.6}$$

Here the order of the terms dropped is stated for the case in which the following expansion of the terms of the outer solution is possible:

$$v_\xi \approx v_{\xi 0}\xi^2 + v_{\xi 1}\xi^3, \quad v_\eta \approx v_{\eta 0}\xi + v_{\eta 1}\xi^2, \quad n \approx n_0 + n_1\xi \quad (\xi \to 0) \tag{5.7}$$

For axisymmetric problems (5.6) takes the form

$$I = \frac{3^{1/3}\pi^{1/3}}{2^{1/3}\Gamma(1+1/3)} n_N \left[\iint_\Gamma d\Gamma\, \psi_0^{1/2} \exp\left(-6k\int_0^\eta \psi_0 \frac{d\eta}{R_1}\right)\right]^{2/3} \lambda^{-2/3} + O(\lambda^{-1}) \tag{5.8}$$

The condition for applicability of (5.6) and (5.8) is

$$k \lesssim k^* (1 - \lambda^{-1/3}\xi_0^{-1}) \tag{5.9}$$

6. Let us see how the results obtained above change in the case in which an attractive force, representable in the vicinity of $\Gamma$ in the form (2.11), acts between the particles and the body. We consider the steady-state problem. The principal terms of the outer expansion, determined from (3.1), in the vicinity of $\Gamma$ will be approximately equal to

$$v_\xi \approx -b_1, \quad v_\eta \approx b_2, \quad n \approx n_* \quad (\xi \to 0) \tag{6.1}$$

Here $b_1$, $b_2$, and $n_*$ are functions of $\eta$. They can be determined easily by a numerical technique. Then we find from the condition of diffusion flux comparable with the convective flux in the transitional zone between the regions $Q_2$ and $Q_1$ that no matter how large $\lambda$

$$\partial n / \partial \zeta \sim 1 \quad \text{for} \quad \zeta = \lambda\xi \tag{6.2}$$

Thus, in the region $Q_1$ (2.3) takes the form

$$\frac{\partial}{\partial \zeta} n v_\xi^* - \frac{\partial^2 n}{\partial \zeta^2} \approx 0 \tag{6.3}$$

The omitted term in this equation is of order $\lambda^{-1}$. The solution of (6.3) must match with the outer solution. Therefore

$$n = G \exp\left(\int_0^\zeta v_\xi^* d\zeta\right) \theta(b_1) + n_* b_1 \int_0 d\zeta' \exp\left(\int_{\zeta'}^\zeta v_\xi^* d\zeta\right) \theta(b_1) + O(\lambda^{-1}) \tag{6.4}$$

Here G is a constant which is determined through the boundary conditions for n on $\Gamma$; $\theta(b_1)$ is a unit step function, equal to 1 for $b_1 > 0$ and 0 for $b_1 > 0$ and $b_1 < 0$. We find from (6.3) that

$$j = n_* b_1 \theta(b_1) + O(\lambda^{-1}) \tag{6.5}$$

Consequently, the particle flux in the direction toward $\Gamma$ in the region $Q_1$ is independent of $\xi$ to within terms of order $\lambda^{-1}$ and, what is particularly important, is independent of the selection of the boundary condition for n on the surface $\Gamma$. It follows from (2.4), (2.5) that in the region $Q_1$ the particles can be considered inertialess if the condition $k \ll \lambda^{-1}$ is satisfied. Let us assume that the numbers k actually are sufficiently small. It is easy to see that in this case the particles can be considered inertialess in the region $Q_2$ as well. Therefore

$$v_\xi \approx \alpha(-a_0 + a_1\xi) + (\alpha a_2 + u_1)\xi^2, \quad v_\eta \approx u_2\xi \quad (\xi \to 0) \tag{6.6}$$

If expansion (6.6) exists, the solution of (2.3) in the steady-state case in the region $Q_1$ can be represented in the form of the series

$$n \approx \sum_{m \geqslant 0} n_m \lambda^{-m} \qquad (\lambda \to \infty) \tag{6.7}$$

Series (6.7) is the more effective the larger $\alpha$. For small $\alpha$ it becomes unsuitable for calculations. In fact, for $\alpha = 0$ it is necessary to use the transformation

$$\xi \to \zeta = \lambda^{1/3}\xi$$

For large $\alpha$ the transformation

$$\xi \to \zeta = \lambda\xi$$

leads to the objective.

Thus we can expect to obtain an asymptotic series which is uniform in $\alpha$ by setting

$$\xi \to \xi = \lambda(\alpha + \lambda^{-2/3})\xi$$
$$n \approx \sum_{m \geqslant 0} n_m \lambda^{-m}(\alpha + \lambda^{-2/3})^{-m} \tag{6.8}$$

An analogous phenomenon occurs whenever the coefficient of the principal term of the asymptotic expansion of u + F in $\xi$ approaches zero.

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